

# On the $\pi\pi$ - scattering lengths in the theory with effective Lagrangian

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## Abstract

Use of the effective Lagrangian incorporating both the scalar and pseudoscalar mesons gives a possibility to calculate the  $\pi\pi$ -scattering lengths without attraction of the ChPT theory.

## 1 Introduction

The question about the numerical quantities of the  $\pi\pi$  -scattering lengths engages the minds of theoreticians and the experimentalists so far. Nowadays, a number of the observed events of  $\pi\pi$  - scattering increased many times, that gives a possibility to verify the different approaches to work out a theory of the low-energy  $\pi\pi$  - interaction. It is clear, that an approach based on the use of the Lagrangian scheme looks as more convenient for deep understanding of the theorems concerning the soft pions. Just in this scheme, it is possible to obtain the precise results of the current algebra even on the "trees" level, that is, on the level of the diagrams without loops [1].

In this paper, this peculiarity of the Lagrangian scheme will be used for the calculation of the  $\pi\pi$  scattering lengths. And besides, the conformity of the properties of the QCD objects and objects of the real world will be ensured also. In QCD, the spinless flavoured objects are  $\bar{q}_R t^a q_L$  and their Hermitian conjugate. These objects are formed by opposite-parity components. For this property to be reproduced in the Lagrangian of real particles, it must be expressed in terms of the matrix

$$U = (\sigma_a + i\pi_a)t_a, \quad (1)$$

where  $t_0 = \sqrt{1/3}$ ,  $t_{1\dots 8} = \sqrt{1/2}$ <sub>1,...,8</sub> and  $\sigma_a$  and  $\pi_a$  are nonets of, respectively, scalar and pseudoscalar mesons. This idea is not new and was used in [2] and

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[3], but we go over directly to the final form of the Lagrangian containing all forms of breakdown of chiral symmetry [4], [5]:

$$L = \frac{1}{2}Tr(\partial U_\mu \partial U_\mu^+) - cTr(UU^+ - A^2 t_0^2)^2 - c\xi(Tr(UU^+ - A^2 t_0^2))^2 + \frac{F_\pi}{2\sqrt{2}}Tr(M(U + U^+) + \Delta L_{PS}^{U(1)}). \quad (2)$$

These forms permit to express all properties of  $\sigma$  - mesons using the properties of  $\pi$  - mesons and the parameters  $R = F_K/F_\pi$  and  $\xi$ . The values of them can be found from the data on the decays  $K, \pi \rightarrow \mu\nu$  and identification of the scalar  $\sigma_\pi$  - meson with the resonance  $a_0(980)$  [4], [5], [6].

In a theory with the broken  $U(3)_L \otimes U(3)_R$  chiral symmetry there are two isosinglet  $\sigma$ -particles having nonzero vacuum expectation values  $\langle \sigma \rangle$ . As a result, the Lagrangian(2) contains the vertices  $\langle \pi\pi|\sigma \rangle$ . Then, the set of the pole diagrams with the intermediate  $\sigma$  - meson appears. The amplitude of the  $\pi\pi \rightarrow \pi\pi$  - scattering acquires the form:

$$T_\sigma = \langle \pi_k(p'_1)\pi_l(p'_2)|\pi_i(p_1)\pi_j(p_2) \rangle = A_\sigma \delta_{ij}\delta_{kl} + B_\sigma \delta_{ik}\delta_{jl} + C_\sigma \delta_{il}\delta_{jk}, \quad (3)$$

where

$$A_\sigma = (s - \mu^2) \sum_{n=1,2} \frac{G_n}{m_{\sigma_n}^2 - s}, \quad B_\sigma = (t - \mu^2) \sum_{n=1,2} \frac{G_n}{m_{\sigma_n}^2 - t}, \quad (4)$$

$$C_\sigma = (u - \mu^2) \sum_{n=1,2} \frac{G_n}{m_{\sigma_n}^2 - u}, \quad \mu \equiv m_\pi,$$

and where

$$s = (p_1 + p_2)^2, \quad t = (p_1 - p'_1)^2, \quad u = (p_1 - p'_2)^2, \quad G_n = \frac{g_n^2}{m_{\sigma_n}^2 - \mu^2}. \quad (5)$$

In the theory specified by the Lagrangian (2) , the following relation holds:

$$\frac{G_1}{m_{\sigma_1}^2 - \mu^2} + \frac{G_2}{m_{\sigma_2}^2 - \mu^2} = \frac{1}{F_\pi^2}, \quad F_\pi = 93 \text{ MeV}. \quad (6)$$

For the case of fixed total isospin of a system of initial pions, the expressions for the amplitudes are given in [7]:

$$T^{(0)} = 3A + B + C. \quad (7)$$

$$T^{(1)} = B - C \quad (8)$$

$$T^{(2)} = B + C. \quad (9)$$

The decomposition of the isotopic amplitudes into amplitudes corresponding to fixed values of the orbital angular momentum is given by

$$T^{(I)} = 32\pi \sum_{l=0}^{\infty} (2l+1) t_l^{(I)}(s) P_l(\cos \theta). \quad (10)$$

It follows from (10) that the partial-wave amplituda  $t^{(I)}$  is:

$$t_l^{(I)}(s) = \frac{1}{64\pi} \int_{-1}^1 T^{(I)} P_l(\cos \theta) d \cos \theta. \quad (11)$$

The scattering lengths arise from the expansion

$$t_l^I(s) m_\pi^{-1} = q^{2l} [a_l^{(I)} + b_l^{(I)} q^2 + O(q^4)], \quad q^2 = \frac{s}{4} - \mu^2. \quad (12)$$

To calculate  $a_0^{(0)}$  and  $a_0^{(2)}$ , we make use of the relations (3,7,9,11). The scattering length  $a_1^{(1)}$  will be considered later, since besides the  $\sigma$  - mesons, the  $\rho$ -mesons also contribute into this scattering length. We begin calculations from  $a^{(2)0}$ , because this scattering length, according to (9) and (7), enters into  $a_0^{(0)}$ .

## 2 The scattering length $a_0^{(2)}$

According to (9)

$$T_\sigma^{(2)} = \sum_{n=1,2} G_n \left( \frac{t - \mu^2}{m_{\sigma_n}^2 - t} + \frac{u - \mu^2}{m_{\sigma_n}^2 - u} \right). \quad (13)$$

On the threshold  $t$  and  $u$  are equal to zero. Using the expressions for masses and coupling constants of the  $\sigma_n$  - mesons [5] and the last numerical data on their values [6], we obtain:

$$T_\sigma^{(2)} = - \left( \frac{2G_1\mu^2}{m_{\sigma_1}^2} + \frac{2G_2\mu^2}{m_{\sigma_2}^2} \right) = -4.3171. \quad (14)$$

According to (11) and (12)

$$a_0^{(2)} = -0.04294 m_\pi^{-1}. \quad (15)$$

This result is in agreement with the result of analysis of the  $K^\pm \rightarrow \pi^+ \pi^- e^\pm \nu$  decay, based on the statistics of 1.13 million decays [8]:

$$a_0^{(2)} = (-0.0432 \pm 0.0086_{stat} \pm 0.0034_{syst} \pm 0.0028_{th}) m_\pi^{-1} \quad (16)$$

### 3 The scattering length $a_0^{(0)}$

In accordance with (7) , the amplitude of  $S$ -wave with the isospin 0 looks as

$$T^{(0)} = \sum_{n=1,2} G_n \left( 3 \frac{s - \mu^2}{m_{\sigma_n}^2 - s} \right) + T_{\sigma}^{(2)}. \quad (17)$$

At the threshold, the first addendum in (17) performs into

$$9\mu^2 \sum_{n=1,2} G_n (m_{\sigma_n}^2 - 4\mu^2)^{-1} = 23.3335. \quad (18)$$

Adding the result (14) for  $T_{\sigma}^{(2)}$ , we come to

$$T_{\sigma}^{(0)} = 19.0164. \quad (19)$$

And the final result for  $a_0^{(0)}$  is:

$$(a_0^{(0)})_{\sigma} = \frac{19.0164}{32\pi m_{\pi}} = 0.18916 m_{\pi}^{-1}. \quad (20)$$

This value agrees with the experimental one:

$$(a_0^{(0)})_{exp} = (0.197 \pm 0.010) m_{\pi}^{-1} \quad (21)$$

obtained from the analysis of all data near threshold of the reaction  $\pi N \rightarrow \pi\pi N$  [9].

The data obtained for  $a_0^{(0)}$  by the collaboration E865 [10] depend from the Models used at analysis. In particular, in the Model A  $a_0^{(0)} = (0.184 \pm 0.010) m_{\pi}^{-1}$ , in the Model B  $a_0^{(0)} = (0.179 \pm 0.033) m_{\pi}^{-1}$  and in the Model C  $a_0^{(0)} = (0.213 \pm 0.013) m_{\pi}^{-1}$ .<sup>2</sup> However, these results appear after taking into account the isospin corrections, also depending from the Models.

Our result (20) does not demand any additional model corrections.

### 4 The scattering length $a_1^{(1)}$

In the used by us theory, besides the  $\sigma$  - mesons, the intermediate  $\rho$  - mesons also give a contribution into  $a_1^{(1)}$ . In the present paper, a nature of the  $\rho$

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<sup>2</sup>See the Table 6 in [8]

- mesons will be not associated with the vector quark current, as it was considered usually, but their nature will be associated with the divergence of vector quark current [11 - 15].

A contibution of the  $\sigma$  - mesons into isovector amplitude is:

$$T_\sigma^{(1)} = \sum_{n=1,2} G_n \left[ \frac{t - \mu^2}{m_{\sigma_n}^2 - t} - \frac{u - \mu^2}{m_{\sigma_n}^2 - u} \right]. \quad (22)$$

According to the relations (11) and (12), the part of scattering length  $a_1^{(1)}$  produced by  $\sigma$  - mesons is:

$$(a_1^{(1)})_\sigma = \frac{1}{24\pi m_\pi} \left[ \frac{g_1^2}{m_{\sigma_1}^4} + \frac{g_2^2}{m_{\sigma_2}^4} \right]. \quad (23)$$

Using the numerical values of  $m_{\sigma_{1,2}}$  and  $g_{1,2}$ , we find:

$$(a_1^{(1)})_\sigma = 0.02744 m_\pi^{(-3)}. \quad (24)$$

The part of the isovector amplitude produced by the intermediate  $\rho$  - mesons looks like the relation (3), but with the differnt A, B and C [15]. Namely,

$$A_\rho = \frac{1}{M_\rho^2} \left( g^2(u) \frac{(s-t)u}{M_\rho^2 - u} + g^2(t) \frac{(s-u)t}{M_\rho^2 - t} \right). \quad (25)$$

$$B_\rho = \frac{1}{M_\rho^2} \left( g^2(s) \frac{(t-u)s}{M_\rho^2 - s} + g^2(u) \frac{(t-s)u}{M_\rho^2 - u} \right). \quad (26)$$

$$C_\rho = \frac{1}{M_\rho^2} \left( g^2(s) \frac{(u-t)s}{M_\rho^2 - s} + g^2(t) \frac{(u-s)t}{M_\rho^2 - t} \right). \quad (27)$$

In accordance with (8)

$$T_\rho^{(1)} = B_\rho - C_\rho. \quad (28)$$

As the explicit form of  $g(x = s, t, u)$  is not specified by our theory, we are forced to resort to the phenomenological relation elaborated in [15]:

$$g(x) = g_\rho \exp \left( 0.7855 \left[ \frac{x}{2M_\rho^2} - \left( \frac{x}{2m_\rho^2} \right)^2 \right] \right), \quad x \leq M_\rho^2. \quad (29)$$

As we are interesting in behavior of the partial wave  $t_1^{(1)}(s)$  near threshold, we obtain the following result:

$$(t_1^{(1)})_{\rho}^{threshold} = \frac{4\mu^2}{3\pi M_{\rho}^2} \left( \frac{2g^2(4\mu^2)}{M_{\rho}^2 - 4\mu^2} + \frac{g^2(0)}{M_{\rho}^2} \right) \quad (30)$$

Using the experimental value of  $g(M_{\rho}^2)=5.9764$  and the formula (29), we find:

$$g(0) = 4.9108, \quad g(4\mu^2) = 5.5520. \quad (31)$$

The part of the scattering length  $a_1^{(1)}$  produced by the intermediate  $\rho$  - mesons turns out to be:

$$(a_1^{(1)})_{\rho} = 0.005918m_{\pi}^{(-3)}. \quad (32)$$

Together with the part (24) produced by the  $\sigma$  -mesons we get:

$$(a_1^{(1)})^{total} = 0.03336m_{\pi}^{-3}. \quad (33)$$

## 5 Conclusion

The requirement of conformity between the properties of the QCD objects and objects of the real world gives rise to necessity of existence of the scalar mesons, that, as it turned out, play the principal role in the low-energy  $\pi\pi$  - interactions. Being the chiral partners of the pseudoscalar mesons, the scalar mesons possess quite definite properties, ascribed by the structure of the Lagrangian (2). Their masses and coupling constants depend only on two parameters:  $R = F_K/F_{\pi}$  and  $\xi$ , the values of which are determined experimentally. The found by us scattering lengths are:

$$a_0^{(0)} = 0.18916m_{\pi}^{-1}, \quad a_0^{(2)} = -0.04294m_{\pi}^{-1}, \quad a_1^{(1)} = 0.03336m_{\pi}^{(-3)}. \quad (34)$$

And they did not require to attract the special models, concerning these scattering lengths. In our theory, the current-algebra prediction [16]:

$$\frac{2a_0^{(0)} - 5a_0^{(2)}}{18\mu^2\pi a_1^{(1)}} = 1 \quad (35)$$

is satisfied to within 1.24%.

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